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Hadron Physics and Transfinite Set Theory

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Known results in transfinite set theory appear to anticipate many aspects of modern particle physics. Extensive and powerful analogies exist between the very curious theorems on "paradoxical" decompositions in transfinite set theory, and hadron physics with its underlying quark theory. The phenomenon of quark confinement is an example of a topic with a natural explanation via the analogies. Further, every observed strong interaction hadron reaction can be envisaged as a paradoxical decomposition or sequence of paradoxical decompositions. The essential role of non-Abelian groups in both hadron physics and paradoxical decompositions is one mathematical link connecting these two areas. The analogies suggest critical roles in physics for transfinite set theory and nonmeasurable sets.

1. TRANSFINITE SET THEORY AND PHYSICS

Transfinite set theory has a peculiar place in mathematical physics. Modern analysis, on which mathematical physics rests substantially, in turn draws on the abstract theory of sets in many ways—e.g., for the structure of the number system, for measure theory, for set theoretic topology, and for other tools. In this sense, set theory underlies physics. In contrast, the objects of concern to transfinite set theory are not generally considered to be the objects encountered in physics. For this one would have to introduce real infinities into physics, and the implications of such are rarely discussed.

This paper explores connections between the objects of physics and the objects of set theory. Section 2 shows that strong and unexpected analogies exist between hadron physics with its underlying quark theory and certain theorems of transfinite set theory on "paradoxical" decompositions of solid objects in R^3. In effect, hadron physics provides a model satisfying these theorems, if we suppose the particles of hadron physics to be composed of transfinitely many constituents, as is the case for their set theoretic counter-

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parts (solid objects in R^3). Section 3 includes a table summarizing the analogies, and notes some mathematical implications from the analogies.

The analogies give directly a large number of known physical results (Section 2), and suggest additional ones testable in principle (Section 3). The quark color label and the phenomenon of quark confinement are examples of topics which have immediate explanations via analogies with the decomposition theorems.

Many mathematicians regard "paradoxical" decompositions as rather troubling and abstract curiosities (Section 3). The physicist, familiar with phenomena such as pair creation, no longer considers these phenomena strange. The aspect of "paradoxical" decompositions probably most interesting to the physicist is that these decompositions simulate, at a geometric level, such familiar phenomena of particle physics. The mathematician's "paradoxes" contain an organizing principle for hadron behavior and link transfinite theory with physical objects and events. These links apparently give another remarkable instance of purely geometrical constructs playing an important role in physical laws. Other instances of such roles are noted by Yang (Yang, 1981).

2. PARADOXICAL DECOMPOSITIONS—ROLE IN PARTICLE PHYSICS

Any respectable hadron and quark description should at least account for (i) hadrons being two-quark or three-quark combinations, (ii) inhibitions against hadrons containing more than three quarks, (iii) hadron–quark properties producing quark confinement, and (iv) distinguishable quarks satisfying the exclusion principle.

This paper shows how these and other features of quark theory can be regarded as immediate consequences of a single known existence theorem of transfinite set theory under a suitable, if speculative, interpretation. Additional theorems of this kind lead to other familiar results of hadron physics.

The theorems used are the astonishing paradoxical decomposition theorems which arise in R^m, $m \geqslant 3$. Decomposition involves starting with some initial bounded solid body, partitioning or cutting that solid into a finite number, n, of pieces, and then reassembling those pieces into some final bounded solid or (disjoint) solids. The pieces are to undergo Euclidean transformations only, and so are one-to-one congruent in the sense of elementary geometry in the initial and final solids. The singular aspect of paradoxical decompositions is that volume of the solids need not be conserved in the decomposition and reassembly, although isometry in the pieces is preserved. For this reason, the theorems are often considered highly nonintuitive (or "paradoxical"). The pieces are generally nonenumerable, nonmeasurable sets to give volume nonconservation; otherwise additivity of measure would be violated. The exotic nonmeasurable sets required need the axiom of choice of set theory for their very intricate formulation.

Decomposition theorems then mimic mathematically physical processes such as particle production and particle annihilation. Theorem 1 shows some of this interesting behavior. The theorem is one of the results known as the "Banach–Tarski paradoxes"[1] (Banach and Tarski, 1924).

> *Theorem 1.* A solid sphere S in R^3 can be cut into a finite number, n, of pieces which can be reassembled into two disjoint solid copies of the sphere S, each copy having the same size as the initial S.

This process can be repeated on one or both copies of S. Consequently, any number of copies of S can be produced from one initial S. The decomposition process is symbolized by "$\sim$" here; as a mathematical operator $\sim$ is symmetric and transitive. Theorem 1 is then $S \sim S + S$; a corollary from symmetry of $\sim$ is $S + S \sim S$. (From Theorem 1 follows the even more startling Theorem 1′: If X, Y are *any* bounded solids, $X \sim Y$. Thus decomposition can produce solid bodies which are bounded but of *arbitrary size and shape*. In particular, Theorem 1′ allows each of the three spheres of Theorem 1 to have a *different* radius.)

The values of n in Theorem 1 can be calculated. von Neumann (v. Neumann, 1928)[2] gave $n = 9$; Sierpinski (Sierpinski, 1945) found $n = 8$. R. Robinson (Robinson, 1947) finally gave $n = 5$, which he also proved is the best possible result (the minimum decomposition).[3] Theorem 2 is the very remarkable Robinson result.

[1] Proofs of the paradoxical decomposition theorems are quite detailed. An essential technical point occurs at an intermediate proof stage. Here one has characterized a nonenumerable set of points, A, as comprised of a nonenumerable number of different sets, each of which is infinite but enumerable. Then use of the axiom of choice of set theory allows exact characterization of A by a single new nonenumerable set, constructed by random selection of just one point from each of the initial nonenumerably many different sets, and a generating rule which is applied enumerably many times to that single new set, to reproduce A. If one accepts the axiom of choice, the proofs of the decomposition theorems are impeccable, no matter how strange the consequences seem. The feature of random selection in the axiom of choice makes the proofs nonconstructive.

[2] This paper gives several new paradoxical decompositions, and most importantly stresses the essential role of the underlying group of motions.

[3] Proofs for any minimal n' appropriate for the general theorem 1′ are unknown, though bounds can always be estimated.

Theorem 2. A solid sphere S can be cut in five pieces; three pieces can be reassembled into one copy of S, the other two pieces into a second copy of S. Pieces within each copy have no congruence relations among themselves, and so the pieces are distinguishable. One congruence relation exists between the pieces of the two copies.

Now speculatively we can interpret in Theorem 2 pieces as quarks, congruence of pieces as indistinguishability of quarks, spheres as hadrons. Theorem 2 then captures the hadron–quark features of the initial paragraph of this section. Two- and three-quark hadrons reflect the minimum decomposition of Theorem 2; the nonminimum decompositions of von Neumann and Sierpinski would allow four-, five-, and six-quark hadrons, so that a "minimum decomposition principle," which warrants sharper characterization, seems generally at work; quark confinement effectively follows from the minimum decomposition of Theorem 2 and the trivial generalization that for no $n > 5$ is a sphere decomposition possible in which any sphere copy contains only one piece; and, the distinguishability of the pieces within the copies is the equivalent of the quark color label. [The standard color assignment giving at any one time one common color (anticolor) between mesons and baryons (antibaryons) also has its exact analog in Theorem 2.]

Let us associate in Theorem 2 the two-piece copy with a meson, the three-piece copy with a baryon. In an obvious notation in which S_2 is a two-piece copy, etc., Theorem 2 reads $S \sim S_2 + S_3$. The copies themselves are subject to Theorem 2, so $S_2 \sim S_2 + S_3$, $S_3 \sim S_2 + S_3$. In particles, we assign standard quantum numbers to the pieces, just as they are assigned to quarks; impose (for brevity of presentation) conservation laws for only charge and baryon number. When particles are not explicitly named, "2" indicates a meson, "3" a baryon; a bar over any particle symbol (e.g., $\bar{3}$) denotes the corresponding antiparticle.

The decomposition $S_3 \sim S_2 + S_3$ then is interpretable as an equivalent particle reaction, and the decomposition operator $\sim$ acts like the particle reaction operator $\rightarrow$; thus $3 \rightarrow 2+3$. Decomposition symmetry ($S_2 + S_3 \sim S_3$) necessarily gives also $2+3 \rightarrow 3$. The other set of equivalent reactions from Theorem 2 ($2 \rightarrow 2+3$, $2+3 \rightarrow 2$) is now forbidden by baryon conservation. Allowed reactions from Theorem 2 then correspond to the large class of Yukawa reactions ($P \rightarrow \pi^+ + N \rightarrow P$; etc.). The interesting feature here is that symmetry of the decomposition operator $\sim$ automatically mandates that the equivalent particle reactions must be able to go in both directions. Microreversibility must hold and the particle reaction operator can be written as $\leftrightarrow$. Symmetry of $\sim$ thus requires for particles precisely the possibility of temporary violations of energy conservation which is permitted by the uncertainty principle of physics, and which is associated with virtual particles.

In subsequent examples, this analog of virtual reactions in the decomposition theorems will be seen as essential in order for decompositions to replicate general hadron reactions. As descriptive models, the decomposition theorems cannot be expected to give quantitative dynamical results, such as the energies involved, without further assumptions. However, in a typical decomposition, such as decomposition of one initial sphere into M pieces from which N copies of that initial sphere are producible—this is always possible from applications of Theorem 1 and the transitivity of $\sim$—it is natural to correlate the energy needed with N and M; and to assume that if the N sphere copies are decomposed and reassembled into the single initial sphere—which is always possible because of symmetry of $\sim$—that energy is recovered. In this view a minimum decomposition principle would have kinship with a minimum energy principle.

The Robinson result is generalized by Theorem 3, due to Mycielski (Mycielski, 1955).

> *Theorem 3.* A solid sphere S can be cut into $2+3(l-1)$ pieces, $1 < l \leqslant 2^{\aleph_0}$, which can be reassembled into l copies of S—one two piece copy and $l-1$ three piece copies.

Theorem 3 allows a decomposition such as: $S_2 \sim S_2 + S_3 + S_3$. This can appear in equivalent particle reaction form as $2 \to 2+3+\bar{3}$. Charge and baryon conservation are then satisfiable in meson transformations. Theorem 3 now permits baryon pairs to be "created" in decompositions—e.g., the $3, \bar{3}$ pair in $2 \to 2+3+\bar{3}$— and "annihilated," since the symmetry of $\sim$ allows also $S_2 + S_3 + S_3 \sim S_2$—i.e., $2+3+\bar{3} \to 2$. In a similar way repeated applications of Theorem 2 allow meson pairs to be created and annihilated —e.g., $S_3 \sim S_2 + S_3$ (Theorem 2); $S_3 \sim S_2 + S_2 + S_3$ (Theorem 2 on the right-hand S_3), which in particle form satisfying conservation laws can be written $3 \to 2+\bar{2}+3$, so that a $2, \bar{2}$ pair appears. This sequence may, e.g., be realized as follows: $P \to \pi^+ + N \to \pi^+ + \pi^- + P$, a π^+, π^- pair appearing.

Theorems 2 and 3 combined give other direct analogies with hadron theory. As one example, from Theorem 3 typically $(S_2 \text{ or } S_3) \sim S_2 + S_3 + S_3 + \cdots$; then Theorem 2 applied to each copy and transitivity of $\sim$ lead with rearrangement very generally to $(S_2 \text{ or } S_3) \sim S_2 + S_2 + S_2 + \cdots + S_3 + S_3 + S_3 + \cdots$. This can always be written as an equivalent particle reaction satisfying conservation laws; for instance, $(2 \text{ or } 3) \leftrightarrow 2+2+2+ \cdots +(2 \text{ or } 3)+3+\bar{3}+3+\bar{3}+ \cdots$. The decomposition theorems thus lead to the virtual hadronic particle clouds (mesons plus baryon–antibaryon pairs) around "bare" hadrons, virtual clouds permitted by quantum mechanics and sensed by experiments.

Another example treats decompositions and equivalent reactions (subject to conservation laws) side-by-side. Thus, Theorem 3 gives $S_2 \sim S_2 + S_3$

$+ S_3$ (e.g., $\pi^0 \rightarrow \pi^0 + P + \bar{P}$); Theorem 2 and symmetry of $\sim$ give $S_2 + S_3 \sim S_3$ ($\pi^0 + P \rightarrow P$); this result and transitivity of $\sim$ give $S_2 \sim S_3 + S_3$ ($\pi^0 \rightarrow P + \bar{P}$). Then, comparing $\pi^0 + P \rightarrow P$ and $\pi^0 \rightarrow P + \bar{P}$ indicates that it is consistent to transpose a particle in a reaction as the corresponding antiparticle. One then also gets consistently $P = \bar{\bar{P}}$; and from this $\pi^0 = \bar{\pi}^0$.

A third example interprets an observed collision reaction such as: $P + P \rightarrow P + P + P + \bar{P}$ via decompositions. Conventionally, a P, $\bar{P}$ pair is here regarded as created from the vacuum. An alternative view uses the decomposition theorems in the following way. The "minimum decomposition" applies Theorem 3 to just one S_3: $S_3 + S_3 \sim S_3 + S_2 + S_3 + S_3 + S_3$ (i.e., $P + P \rightarrow P + \pi^0 + P + P + \bar{P}$); symmetry of $\sim$ in Theorem 2 gives $S_2 + S_3 \sim S_3$ ($\pi^0 + P \rightarrow P$); finally, from transitivity of $\sim$, $S_3 + S_3 \sim S_3 + S_3 + S_3 + S_3$ ($P + P \rightarrow P + P + P + \bar{P}$). Another (nonminimum) decomposition would apply Theorem 2 to one S_3, Theorem 3 to the other S_3 to get $S_3 + S_3 \sim S_2 + S_3 + S_2 + S_3 + S_3 + S_3$ ($P + P \rightarrow \pi^0 + P + \pi^0 + P + P + \bar{P}$), where now two π^0 could form and disappear; and so on, more and/or different particles forming and disappearing.

A final example is the meson reaction: $\omega \rightarrow \pi^0 + \pi^+ + \pi^-$, seen through a sequence of intermediate virtual reactions: $S_2 \sim S_2 + S_3 + S_3$, (e.g., $\omega \rightarrow \pi^0 + P + \bar{P}$, Theorem 3); $S_2 \sim S_2 + S_2 + S_3 + S_2 + S_3$ ($\omega \rightarrow \pi^0 + \pi^+ + N + \pi^- + \bar{N}$, Theorem 2 on the S_3); $S_2 \sim S_2 + S_2 + S_2 + S_3 + S_3$ ($\omega \rightarrow \pi^0 + \pi^+ + \pi^- + N + \bar{N}$, rearrangement); finally $S_2 \sim S_2 + S_2 + S_2$ ($\omega \rightarrow \pi^0 + \pi^+ + \pi^-$, Theorem 3 and symmetry of $\sim$ giving $S_2 + S_3 + S_3 \sim S_2$ or $\pi^- + N + \bar{N} \rightarrow \pi^-$).

These examples show how each observed strong interaction hadron reaction can be interpreted as a paradoxical decomposition or as well-defined sequences of paradoxical decompositions.

3. SUMMARY AND ADDITIONAL IMPLICATIONS

We have curious and remarkable outcomes. Theorems 2 and 3, and properties of $\sim$, mirror established general features of hadron physics. A substantially interesting portion of particle physics is simulated by the paradoxical decomposition theorems.

Some principal analogies of Section 2 are summarized by Table I.

No known experiments seem inconsistent with the speculative interpretation in Theorem 2 of pieces as quarks and spheres as hadrons, and some further possible consequences are suggested. For example, rms charge radii for protons and pions are different; but those data reflect measurements of physical hadrons with their extended virtual clouds. Experiments indicating the size, and perhaps size equivalence in some sense, of all "bare" hadrons

TABLE I. Analogies—Set Theory and Physics

Aspects of decomposition theorems:	link with these aspects of hadron physics
Decomposition operator ~	Particle reaction operator →
Symmetry of ~	Reversibility of → (i.e., ↔)
Transitivity of ~	Reaction chains
Theorem 1	Production of many hadrons from single hadrons
Generate copies from initial body	Draw particles from physical vacuum (particle creation)
Theorem 2 (Robinson minimum decomposition)	2-quark, 3-quark hadrons
"Minimum" decomposition principle	Inhibits hadrons with $\geqslant 4$ quarks
Theorem 2, and no possible $n-1, 1$ split	Quark confinement
Distinguishability of pieces in Theorem 2	Color label
One congruence between Theorem 2 copies	One common meson/baryon color label
Theorem 2 plus symmetry of ~	Virtual reactions are required
Theorem 3	Baryon pair creation/annihilation
Decompositions with many copies	Virtual particle clouds
Theorems 2, 3, properties of ~	Observed hadron reactions
Multiple paths for decompositions	Multiple reaction paths
Number of pieces order decomposition	Reaction paths with differing probabilities
(Minimum decomposition path)	(Favored—most likely—reaction path?)
Non-Abelian groups underlie theory	Non-Abelian groups underlie theory
Existence of nonmeasurable sets	Particle numbers (or volume) not conserved
All bounded solids generated from single initial solid (Theorems 1′, 2 and 3)	All particles generatable from one particle ("bootstrap hypothesis")

would be interesting. Size equivalence would be a sufficient condition for the interpretation of Theorem 2 as a purely physical theorem (but not a necessary condition, because of Theorem 1′).

The picture of hadron reactions provided by decomposition theorems has interesting subdetail. In Theorem 2 the initial sphere must first be separated into five pieces which are then reassembled into the two sphere copies—the two sphere copies cannot simply be extracted directly, as it were, from the initial sphere. This picture is consistent with the notion that hadron reactions have intermediate stages with "unassembled" quarks. In Theorem 2 all pieces but one have a nonenumerable number of points. The exception is that one piece (i.e., one quark) in the three-piece copy (i.e., in the "baryon") consists of a single point only. This very strongly suggests the remaining pieces (i.e., quarks) physically are composite entities with structure—a prediction in principle amenable to test at very high energies (few to many TeV center-of-mass energy).

How one should ultimately regard the significance of these analogies is unclear. Essential theoretical underpinnings for both hadron physics and paradoxical decompositions are based on non-Abelian group properties (for the paradoxical decompositions, the group theoretic issues are clearly treated in the referenced von Neumann paper). Existence of analogies therefore may possibly be not too surprising. But the analogies could still simply reflect fantastic coincidences; or, they could suggest some unsuspected aspects of physical reality.

A conclusion that paradoxical decompositions have counterparts in the physical world immediately touches mathematical issues and enhances the probability that axioms of set theory are "true." Bounded nonenumerable nonmeasurable sets are necessary for the decomposition theorems. Such sets are constructed using the axiom of choice (AC), and cannot be constructed without AC (Solovay, 1970). But the decomposition theorems are often seen as so bizarre as to cause in mathematicians lingering doubts about AC. Weaker, very plausible choice principles exist which do not yield nonmeasurable sets, but do permit other agreeable consequences of AC (Moore, 1982). Existence of the hadron-related analogies selects AC over any other choice principle not powerful enough to formulate nonmeasurable sets. Only AC in this way leads to physical consequences which accord with everyday experiences of hadron physicists. We therefore find support for a mathematical axiom, AC, from *physical* arguments; and a direct physical role is for the first time found for *nonmeasurable* sets.

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